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Scale-invariance in three-dimensional isotropic turbulence: a paradox and its resolution

David McComb

School of Physics, University of Edinburgh, Edinburgh EH9 3JZ, UK

E-mail: wdm@ph.ed.ac.uk

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Abstract

If the Reynolds number is large enough, turbulence is expected to exhibit scale invariance in an intermediate ('inertial') range of wave numbers, as shown by power-law behaviour of the energy spectrum and also by a constant rate of energy transfer through wave number. However, although it has long been known that the first of these is true, there has been little recognition of the fact that, if the second is to hold, then there is a contradiction between the definition of the energy flux (as the integral of the transfer spectrum) and the observed behaviour of the transfer spectrum itself. This is because the transfer spectrum $T(k)$ is invariably found to have a zero crossing at a single point (at k_0, say), implying that the corresponding energy flux cannot have an extended plateau but must instead have a maximum value at $k = k_0$. We outline the resulting paradox and note that it may be resolved by the observation that the symmetry of the triadic interactions means that $T(k)$ is not the relevant transfer term in determining the energy flux. Instead the relevant term is a filtered/partitioned version, herein denoted by $T^{+-}(k|k_c)$, where $k = k_c$ is the cut-off wave number for low/high-pass filtering. It is known from studies of spectral subgrid transfer that $T^{+-}(k|k_c)$ is zero over an extended range of wave numbers. As this is the case for quite modest Reynolds numbers, it not only resolves the paradox, but may also shed some light on the 'embarrassment of success' of the Kolmogorov theory.

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1. Introduction

In this paper, we shall consider an aspect of the Kolmogorov theory [1, 2] (K41) which does not appear to have received much attention. For many years K41 has had a question mark hanging over its status as a theory of inertial-range turbulence: for a discussion and references see the review by Sreenivasan [3]. Here we will put forward an analysis based on the concept of scale invariance. While we accept the approximations inherent in K41, we do not introduce any new approximations here. All steps taken by us are exact and rigorous.

In order to simplify the analysis and exclude some extraneous considerations, we shall focus on isotropic turbulence. This restricts the concept of *universality* to mean *independence of spectra from the way in which the isotropic turbulence is generated*. We shall consider the wider concept, involving different flow fields, in another paper.

2. Scale invariance

The term *scale invariance* comes from the theory of critical phenomena, but the concept itself has been recognized in turbulence theory for a very long time. More than half a century ago, Batchelor (see [4] for the second edition of this classic monograph), when referring to wave number ranges in turbulence at very large Reynolds numbers, commented

> '. . . the only connection between the equilibrium range and the remainder of the turbulence lies in the transfer of energy at a rate ε.'

In other words, the inertial range of wave numbers is characterized by a constant energy flux at a rate equal to the viscous dissipation. This view arises from the Richardson–Kolmororov picture of a local cascade, supported originally (as Batchelor noted) by the experimental measurements of Townsend, indicating a separation of energy-containing and viscous ranges, as early as 1938.

Following Batchelor, we may develop this idea in the context of the (by now) well-known spectral energy balance equation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\nu_0 k^2\right) E(k,t) = T(k,t), \tag{1}$$

where $E(k,t)$ is the energy spectrum, $T(k,t)$ is the energy transfer spectrum and ν_0 is the kinematic viscosity.

Now let us integrate each term of (1) with respect to wave number, from zero up to some arbitrarily chosen wave number κ:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\kappa \mathrm{d}k\, E(k,t) = \int_0^\kappa \mathrm{d}k\, T(k,t) - 2\nu_0 \int_0^\kappa \mathrm{d}k\, k^2 E(k,t). \tag{2}$$

The energy transfer spectrum may be written as

$$T(k,t) = \int_0^\infty \mathrm{d}j\, S(k,j;t), \tag{3}$$

where, as is well known, $S(k,j;t)$ can be expressed in terms of the triple moment. Its antisymmetry under interchange of k and j guarantees energy conservation in the form

$$\int_0^\infty \mathrm{d}k\, T(k,t) = 0. \tag{4}$$

With some use of the antisymmetry of S, along with equation (4), equation (2) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\kappa \mathrm{d}k\, E(k,t) = -\int_\kappa^\infty \mathrm{d}k \int_0^\kappa \mathrm{d}j\, S(k,j;t) - 2\nu_0 \int_0^\kappa \mathrm{d}k\, k^2 E(k,t). \tag{5}$$

In this familiar form[1], the integral of the transfer term is readily interpreted as the net flux of energy from wave numbers less than κ to those greater than κ, at any time t.

[1] This is identical, notational differences apart, to equation (5.5.16) in [4].

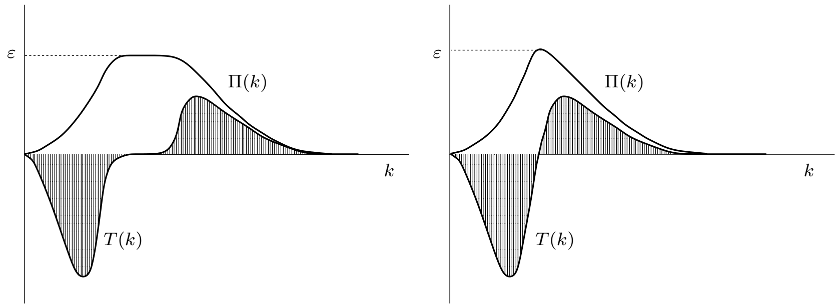

Figure 1. Schematic views, of the energy flux and energy transport spectrum as functions of wave number, which illustrate the paradox. On the left (after Davidson [7]), we show the usual criterion of constant flux for the inertial range, and the corresponding curve for $T(k) = \partial\Pi/\partial k$, as calculated from (6). But, in practice, the transfer spectrum is never found to take this shape, and on the right we show a realistic $T(k)$, with the flux calculated from equation (6).

It is convenient to introduce a specific symbol Π for this energy flux, thus

$$\Pi(\kappa, t) = \int_{\kappa}^{\infty} \mathrm{d}k\, T(k, t) = -\int_{0}^{\kappa} \mathrm{d}k T(k, t), \tag{6}$$

where the second equality follows from (4).

The inertial range of wave numbers is defined as being where the time derivative and the viscous term are negligible. Hence, from equation (1), it follows that the criterion for an inertial range of wave numbers can be taken as the vanishing of the transfer spectrum; and, from equation (6), the constancy of the flux. In other words, for wave numbers κ *in the inertial range* we have

$$T(\kappa, t) = 0 \qquad \text{and} \qquad \Pi(\kappa, t) = \varepsilon. \tag{7}$$

These criteria have proved very influential. However, strictly, they also require stationarity, whereas most experimental work on isotropic turbulence relies on freely decaying turbulence and therefore can only be approximately stationary for some range of wave numbers (i.e. *local stationarity*).

Scale invariance, over a range of wave numbers, can be summed up as the observation that the energy spectrum takes the form of a power law (which is in itself scale free) and that there is a constant rate of energy transfer, which must necessarily be equal to the rate of energy dissipation. In practice, the second criterion of equation (7) is widely used to identify the inertial range. For example, the books by Leslie [5], McComb [6], and Davidson [7] all follow Kraichnan [8], and cite the criterion $\Pi = \varepsilon$; as does work by, for instance, Bowman [9], Thacker [10] and Falkovich [11].

3. The paradox

There are two inertial-range criteria in (7), and, by elementary calculus, they seem to be equivalent. Thus the criterion $\Pi(\kappa) = \varepsilon$ for inertial-range wave numbers $k_{\mathrm{bot}} \leqslant \kappa \leqslant k_{\mathrm{top}}$ implies $T(\kappa) = 0$ for the same range of wave numbers. This fact is illustrated rather nicely in figure 8.10 of the book by Davidson [7] which is a schematic plot of flux and transfer spectrum against wave number[2]. We present a version of this figure here as the left-hand side of figure 1. It shows an extended region where the flux is constant and also the transfer spectrum is zero. This makes an appealingly simple picture of spectral energy transfers but unfortunately it is wrong. The transfer spectrum always passes through zero at a single point (from now on, we shall refer to this as a *single zero crossing*): it has never been found to behave as shown here. This point is of such importance that we shall briefly summarize the experimental situation as follows.

The discovery of this property of $T(k)$ came about when Uberoi [12] made the first experimental determination of the transfer spectrum. He used the energy balance equation (as given by equation (1)), along with an assumption of isotropy, to determine $T(k, t)$ from measurements of the time derivative of the energy spectrum and the dissipation term. In the process, he found only a single zero crossing in each transfer spectrum. However, he pointed out that, for a rigorous inertial range, both the rate of change of the energy spectrum and the rate of viscous dissipation should be negligible and hence the first criterion of equation (7) should exactly hold. In order to illustrate this view, he plotted a schematic energy balance (see figure 22 in [12]) which associated a '$-5/3$' power-law region of energy spectrum with an extended range over which the transfer spectrum is zero.

Later, extensive investigations confirmed that the transfer spectrum always has a single zero crossing [13, 14] and pragmatic, approximate procedures were introduced to allow the inertial range to be identified from the behaviour of the transfer spectrum [15]. For a discussion of this topic, see [16].

As the left-hand and right-hand sides of figure 1 are mutually exclusive, the question then arises: which form of the energy flux is observed in practice? The left-hand version of Π corresponds to the hypothesis of scale invariance, whereas the right-hand form corresponds to an experimentally realistic form of transfer spectrum taken in conjunction with equation (6).

Unfortunately, the energy flux does not seem to have received much attention from experimentalists. However, we do have some evidence from a forced DNS, where Young [17] plotted Π/ε against wave number and found it to be equal to unity (within experimental error) over the range of wave numbers for which the energy spectrum was judged to have an inertial range. On the other hand, he found that the corresponding $T(k)$ had a single zero crossing, which means that the first part of equation (7) does not hold for an extended range of wave numbers and hence neither can the second part. We show Young's results for the energy flux schematically on the left of figure 2. A paradoxical result indeed! (And see also figures 8–10 of [18]. This is a more complicated situation with broad-band forcing of a DNS, but it is nevertheless possible to discern the behaviour reported by Young.)

The paradox may now be stated as follows. The twin manifestations of scale-invariance are the power-law for the energy spectrum and the constancy of the energy flux with respect to scale or wave number. The power-law is widely observed and its existence is beyond doubt: it is an empirical fact. However, it also appears to be an empirical fact that the transfer spectrum has a single zero crossing and this is a *qualitative* empirical fact which makes it impossible for the energy flux, as defined by equation (6), to be a constant. Yet, results from numerical simulations indicate that it is a constant!

Accordingly, there is an apparent contradiction between these two empirical facts and we are faced with a paradox.

[2] The first printing of McComb's book [6] also illustrated this situation (see figure 2.5) by using a calculation of a particular closure (the LET theory). However, the plot showing an extended region with $T(k) = 0$ was due to an error in the calculation and this was rectified in the second printing.

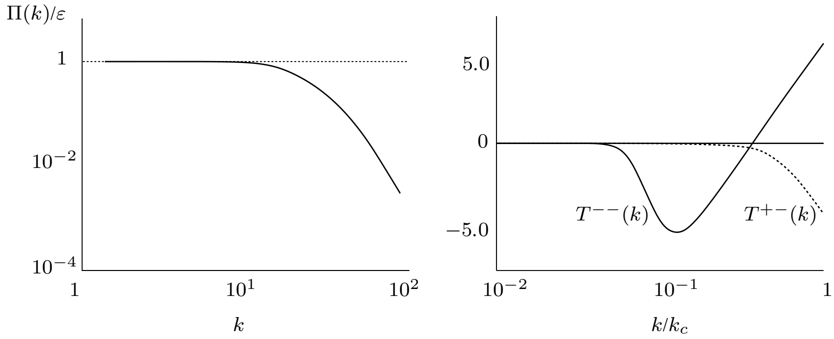

Figure 2. On the left, constant energy flux in the inertial range from the numerical simulation of Young [17]. On the right, a schematic view of the filtered, partitioned transfer spectra as obtained by Zhou and Vahala [19] in a numerical investigation of spectral large eddy simulation and subgrid transfer.

4. Resolution of the paradox

Figure 1 presents a contradiction: on the left-hand side, the correct flux but the wrong transfer spectrum; on the right-hand side the wrong flux but the correct transfer spectrum. The contradiction is entirely dependent on our use of equation (6) and so equation (6) has to go! The clue to how we find a replacement lies in Batchelor's form of the flux balance, that is equation (5).

So, let us consider again equation (5) for the transfer of energy from low wave numbers to high. Now we wish to draw attention to the fact that, although the first term on the right-hand side correctly represents the integral over wave number k of the transfer spectrum from zero up to κ, nevertheless the integrand is not actually $T(k)$ (from now on, we shall suppress time arguments in the interests of conciseness). In fact the integrand represents *some part of* $T(k)$, because the internal integration with respect to the dummy variable j has been truncated at $j = \kappa$.

In order to clarify this situation, it will be found helpful to introduce low- and high-pass filtering operations, based on a cut-off wave number $k = k_c$, on the Fourier components of the velocity field. These operations are familiar from the study of spectral mode elimination in the context of large-eddy simulation and its associated subgrid modelling: see, for example, [20] and references therein. We are thus led to introduce transfer spectra which have been filtered with respect to k and which have had their integration over j partitioned at the filter cut-off, i.e. $j = k_c$.

Beginning with the Heaviside unit step function, defined by

$$H(x) = 1 \qquad \text{for} \qquad x > 0; \tag{8}$$

$$\qquad = 0 \qquad \text{for} \qquad x < 0, \tag{9}$$

we may define low-pass and high-pass filter functions, thus

$$\theta^-(x) = 1 - H(x), \tag{10}$$

and

$$\theta^+(x) = H(x). \tag{11}$$

We may then decompose the transfer spectrum, as given by (3), into four constituent parts,

$$T^{--}(k|k_c) = \theta^-(k-k_c)\int_0^{k_c} \mathrm{d}j\, S(k,j); \tag{12}$$

$$T^{-+}(k|k_c) = \theta^-(k-k_c)\int_{k_c}^{\infty} \mathrm{d}j\, S(k,j); \tag{13}$$

$$T^{+-}(k|k_c) = \theta^+(k-k_c)\int_0^{k_c} \mathrm{d}j\, S(k,j); \tag{14}$$

and

$$T^{++}(k|k_c) = \theta^+(k-k_c)\int_{k_c}^{\infty} \mathrm{d}j\, S(k,j), \tag{15}$$

such that the overall requirement of energy conservation is satisfied

$$\int_0^{\infty} \mathrm{d}k[T^{--}(k|k_c) + T^{-+}(k|k_c) + T^{+-}(k|k_c) + T^{++}(k|k_c)] = 0. \tag{16}$$

It is readily verified that the individual filtered/partitioned transfer spectra have the following properties:

$$\int_0^{k_c} \mathrm{d}k\, T^{--}(k|k_c) = 0; \tag{17}$$

$$\int_0^{k_c} \mathrm{d}k\, T^{-+}(k|k_c) = -\Pi(k_c); \tag{18}$$

$$\int_{k_c}^{\infty} \mathrm{d}k\, T^{+-}(k|k_c) = \Pi(k_c); \tag{19}$$

and

$$\int_{k_c}^{\infty} \mathrm{d}k\, T^{++}(k|k_c) = 0. \tag{20}$$

Equation (2) may be rewritten in terms of the filtered/partitioned transfer spectrum as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{k_c} \mathrm{d}k\, E(k,t) = -\int_{k_c}^{\infty} \mathrm{d}k\, T^{+-}(k|k_c) - 2\nu_0\int_0^{k_c} \mathrm{d}k\, k^2 E(k,t). \tag{21}$$

We note from equation (17) that $T^{--}(k|k_c)$ is conservative on the interval $[0, k_c]$, and hence does not appear in (21), while $T^{-+}(k|k_c)$ has been replaced by $-T^{+-}(k|k_c)$, using (18) and (19).

Evidently this reformulation offers a possibility of resolving the paradox. But, to be sure about this, we need to know how $T^{+-}(k|k_c)$ behaves as a function of wave number. Fortunately, filtered and partitioned transfer spectra have been measured, using DNS, in the context of spectral large-eddy simulation. In particular, Zhou and Vahala [19] found that the resolvable-scale energy transfer spectrum $T^{<<}(k)$ (i.e. $T^{--}(k|k_c)$ in our notation) is conservative on the interval $0 \leqslant k \leqslant k_c$, in agreement with our equation (17); while the resolvable-subgrid transfer spectrum (i.e. our $T^{-+}(k|k_c)$) is zero over a range of wave numbers. We illustrate these results in the right-hand side of figure 2. Similar behaviour has also been found in the more detailed investigation by McComb and Young [21].

5. Discussion and conclusions

As we have seen, the basic elements of the paradox are known, but the apparent contradiction does not appear to have been recognized. This seems to be particularly true for theorists, who do not appear to have realized that the fact that the transfer spectrum has a sharp zero crossing rules out the idea of constant flux, as defined by equation (6), over a range of wave numbers.

We have also seen that the paradox can be resolved by studying the behaviour of $T^{--}(k|k_c)$, as defined by equation (12), and $T^{-+}(k|k_c)$, as defined by equation (13), rather than just $T(k)$ itself. However, just because a paradox has been resolved does not mean that it then goes away. It is still an apparent contradiction and hence still a paradox.

Of course, experimentalists, who do not have access to partitioned versions of the transfer spectrum, will still find pragmatic procedures, such as the Lumley criterion for the inertial range [15], useful. However, those working with DNS or analytical theory, can avoid the paradox by changing their definition of energy fluxes, from those given by (6), to the forms[3]

$$\Pi(\kappa, t) = \int_{\kappa}^{\infty} \mathrm{d}k\, T^{+-}(k|\kappa, t) = -\int_{0}^{\kappa} \mathrm{d}k\, T^{-+}(k|\kappa, t), \tag{22}$$

where $T^{+-}(k|\kappa, t)$ is defined by (14) and $T^{-+}(k|\kappa, t)$ by (13). This is equivalent to (6), but, unlike it, avoids the paradox.

Lastly, the present work may offer some new support to the K41 picture, because a consideration of $T^{+-}(k|\kappa)$, rather than the total (but not relevant!) transfer spectrum may make the K41 result seem less unlikely (e.g. see the discussion by Kraichnan [23]) and hence reduce its 'embarrassment of success'!

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[3] We should mention that these forms are exactly equivalent to Kraichan's original definition of what he called the *transport power* [8]. In later work [22], his definition of the transport power was equivalent to equation (6) in the present paper.

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